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# BOUNDS ON TRANSIENT TEMPERATURE DISTRIBUTION DUE TO A BURIED CYLINDRICAL HEAT SOURCE

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**Abstract**—The theory of differential inequalities is applied to determine upper and lower bounds on the transient temperature distribution in the region surrounding a buried cylindrical heat source of circular cross section. Solutions are obtained for a convection boundary condition on the cylindrical surface and a constant temperature on the free surface with the depth as a parameter. Use of the approximate solutions for engineering estimates of the transient heat transfer from buried pipelines is briefly discussed.

## NOMENCLATURE

$A_n$, coefficients of expansion of $\bar{T}$;
$a$, non negative function out;
$B$, boundary operator;
$c$, length scale for buried pipe;
$D$, spatial domain;
$d$, depth of buried pipe;
$g$, geometric coefficient;
$\bar{g}$, upper bound on $g$;
$\underline{g}$, lower bound on $g$;
$h$, convective heat-transfer coefficient;
$\bar{h}$, upper bound on $h$;
$\underline{h}$, lower bound on $h$;
$h_{\alpha\alpha}$, metric coefficient;
$k$, thermal conductivity;
$p$, parameter in solution $\underline{T}$;
$q$, parameter in solution $\bar{T}$;
$r$, independent variable related to $\alpha$;
$R$, thermal resistance;
$T$, temperature distribution;
$\bar{T}$, upper bound on $T$;
$\underline{T}$, lower bound on $T$;
$T_0$, temperature of heat source;
$t$, time;
$u$, continuous function in $\Omega$;
$u_0$, parameter in $\bar{T}$;
$v$, continuous function in $\Omega$;
$w$, continuous function in $\Omega$;
$x$, rectangular coordinate;
$y$, rectangular coordinate.

Greek symbols

$\alpha$, bipolar coordinate;
$\alpha_0$, constant defining circular cylinder surface;
$\beta$, bipolar coordinate;
$\kappa$, thermal diffusivity;
$\bar{\kappa}$, upper bound on $\kappa$;
$\underline{\kappa}$, lower bound on $\kappa$;
$\bar{\lambda}_n^2$, eigenvalues of $\bar{T}$;
$\phi$, function of $\alpha$ and $t$;
$\rho$, pipe radius;
$\Omega$, domain;
$\partial\Omega$, boundary of $\Omega$.

## 1. INTRODUCTION

HEAT conduction in the region surrounding a cylindrical heat source is of considerable interest because of its application to district heating, pipeline transmission and also underground electrical powerline transmission. Although exact steady-state solutions are available [1, 2] an exact analytic treatment of the transient problem has not yet been derived. Approximate transient solutions were given by Ioff [3, 4] for simple Dirichlet boundary conditions but they are in the form of integrals with oscillatory integrands which are difficult to compute numerically. Also in obtaining error estimates Ioff [4] compared his solution in the steady state with a second steady-state solution taken to be exact. However, it can be shown that the second solution is not exact, thereby invalidating the error estimate.

In the present analysis two approximate solutions to the transient heat equation are given for the region between a plane at constant temperature and a cylinder initially at the same temperature. It is assumed that the cylinder is suddenly filled with a fluid of different constant temperature and that the transfer of heat from the fluid to the region surrounding the cylinder may be described using a constant heat-transfer coefficient at the cylinder surface. These approximate solutions are shown to be bounds on the unknown exact solutions through the use of differential inequalities as previously discussed in Adams [5] and more recently in Sadhal and Martin [6].

A theorem relating approximate solutions to bounds on unknown exact solutions for parabolic operators with mixed boundary conditions is stated without mathematical proof in the following section. (The interested reader should see Walter [7] for rigorous details and also Sadhal [8] for a physical

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argument supporting the theorem.) Further sections describe the exact mathematical formulation and the simplifications used to obtain the approximate solutions. These simplifications are shown to lead to differential inequalities which satisfy the requirements of the discussed theorem. Thus the two derived solutions are shown to be bounds on the solution to the exact mathematical formulation.

## 2. LEMMA 1 [7]

Let $P$ be a parabolic differential operator defined in the domain $\Omega$ and $B$ be a boundary operator defined on the boundary $\partial\Omega$ by

$$B[w] = \begin{cases} \left(\dfrac{\partial w}{\partial n} + aw\right)\Big|_{\partial\Omega_1} \\ w|_{\partial\Omega_2} \end{cases} \tag{1}$$

where $\partial\Omega = \partial\Omega_1 U \partial\Omega_2$, $w$ is a continuous function in $\Omega$, $a$ is a non-negative function in $\partial\Omega_1$ and $\partial/\partial n$ is the outward normal derivative.

Let $u$ and $v$ be continuous functions in $\Omega$ such that $P[u]$, $P[v]$, $B[u]$ and $B[v]$ exist.

If

$$P[u] \leqslant P[v] \text{ and } B[u] \leqslant B[v] \tag{2}$$

then

$$u \leqslant v. \tag{3}$$

Here $\partial\Omega$ is defined as the region over which the boundary conditions are normally specified. For the case of the heat equation

$$P \equiv \frac{1}{\kappa}\frac{\partial}{\partial t} - \nabla^2, \tag{4}$$

$$\Omega = Dx\{t > 0\}, \tag{5}$$

and

$$\partial\Omega = \partial Dx\{t > 0\} U Dx\{t = 0\}. \tag{6}$$

## 3. EXACT MATHEMATICAL FORMULATION

The buried-pipe or cable geometry shown in Fig. 1 can be transformed into the more convenient two-dimensional bipolar coordinates by using

$$x + iy = c\coth[\tfrac{1}{2}(\alpha + i\beta)], \tag{7}$$

$$-\infty < \alpha < \infty, \quad -\pi < \beta < \pi$$

(see Fig. 2 for illustration). In these coordinates the cylindrical surface $\alpha = \alpha_0$ represents a pipe of radius $\rho = c\,\mathrm{csch}\,\alpha_0$ buried a depth $d = c\coth\alpha_0$. The two-dimensional heat equation can then be written as

$$\kappa^{-1}(\alpha,\beta)c^2 g(\alpha,\beta)\frac{\partial T^*}{\partial t^*}(\alpha,\beta,t^*) - \left[\frac{\partial^2 T^*}{\partial \alpha^2}(\alpha,\beta,t^*) + \frac{\partial^2 T^*}{\partial \beta^2}(\alpha,\beta,t^*)\right] = 0 \tag{8}$$

where $T^*$ is the temperature, $\kappa$ is the thermal diffusivity, $t^*$ is time and

$$g(\alpha,\beta) = (\cosh\alpha - \cos\beta)^{-2}. \tag{9}$$

The initial temperature is assumed to be zero everywhere, i.e.

$$T^*(\alpha,\beta,0) = 0 \tag{10}$$

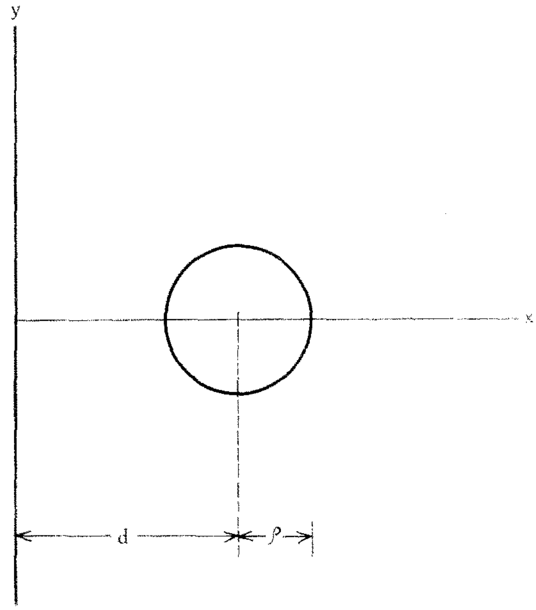


FIG. 1. Definition sketch for buried cylinder.

and the following boundary conditions are applied on the plane and cylindrical surfaces.

$$T^*(0,\beta,t^*) = 0 \tag{11}$$

$$\frac{-k}{h_{\alpha_0} c}\frac{\partial T^*}{\partial \alpha} = h(T^* - T_0^*)|_{\alpha=\alpha_0}. \tag{12}$$

Here $k$ is the thermal conductivity, $h$ is the heat-transfer coefficient, $T_0^*$ is the temperature of the cylindrical heat source and

$$h_{\alpha_0} = (\cosh\alpha_0 - \cos\beta)^{-1}. \tag{13}$$

In addition, continuous temperature and heat flux require that

$$T^*(\alpha,\beta',t^*) = T^*(\alpha,\beta'+2\pi,t^*) \tag{14}$$

and

$$\frac{\partial T^*}{\partial \beta}(\alpha,\beta',t^*) = \frac{\partial T^*}{\partial \beta}(\alpha,\beta'+2\pi,t^*). \tag{15}$$

By introducing the dimensionless variables $T = T^*/T_0^*$ and $t = t^*\kappa_0/c^2$ in terms of the temperature scale $T_0^*$, the length scale $c$ and an appropriate diffusivity $\kappa_0$ the system consisting of (8) and (10)–(15) becomes

$$g\frac{\kappa_0}{\kappa}\frac{\partial T}{\partial t} - \left(\frac{\partial^2 T}{\partial \alpha^2} + \frac{\partial^2 T}{\partial \beta^2}\right) = 0 \tag{8a}$$

$$T(\alpha,\beta,0) = 0 \tag{10a}$$

$$T(0,\beta,t) = 0 \tag{11a}$$

$$-\frac{\partial T}{\partial \alpha} = \frac{hc}{k}\cdot h_{\alpha_0}(T-1). \tag{12a}$$

Even if the diffusivity $\kappa$ is assumed to be constant, an exact solution of this problem cannot be found because of the coefficients $g(\alpha,\beta)$ and $h_{\alpha_0}(\beta)$. Therefore, it is desirable to approximate $g(\alpha,\beta)/\kappa(\alpha,\beta)$ by a function of $\alpha$ alone, say $f(\alpha)$, and $h_{\alpha_0}$ by a constant since doing so

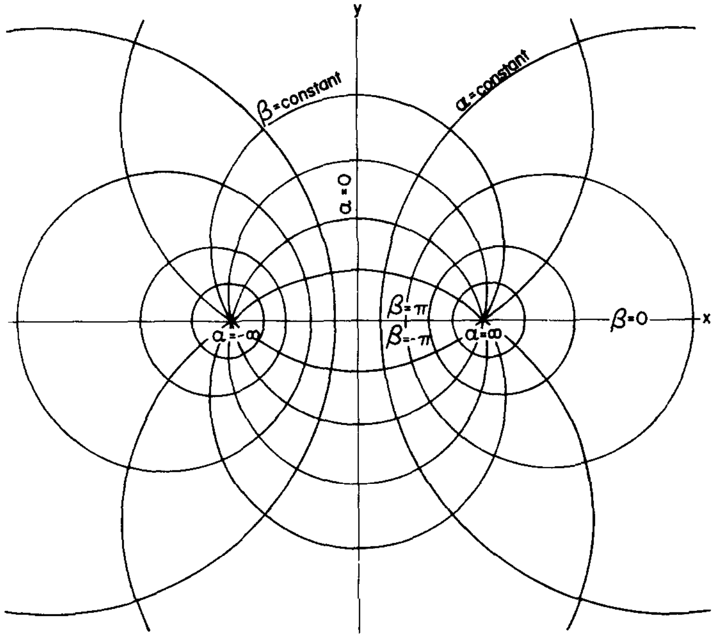


FIG. 2. Bicylindrical coordinate system.

leads to a solution which is independent of $\beta$. The key to finding useful upper and lower bounds on the exact solution $T$ lies in the selection of suitable approximations which bracket the troublesome coefficients $g(\alpha,\beta)/\kappa(\alpha,\beta)$ and $h_{\alpha_0}(\beta)$. In the following section such approximations are presented and are justified using the theory of differential inequalities.

#### 4. APPROXIMATIONS LEADING TO UPPER AND LOWER BOUNDS ON THE TEMPERATURE

It is possible to construct a function $\underline{\kappa}(\alpha,\beta)$ such that

$$\underline{\kappa}(\alpha,\beta) = \underline{\kappa}_0/(\cosh\alpha-\cos\beta)^2\bar{f}(\alpha) \leqslant \kappa(\alpha,\beta) \quad (16)$$

where $\underline{\kappa}_0$ is the minimum of $\kappa(\alpha,\beta)$ and serves as the scale $\kappa_0$ in (8a) and in $t$. If in (8a) $\kappa$ is replaced by $\underline{\kappa}$ then

$$\bar{f}(\alpha)\frac{\partial \underline{T}}{\partial t} - \frac{\partial^2 \underline{T}}{\partial \alpha^2} = 0 \quad (17)$$

where

$$\bar{f}(\alpha) \geqslant g(\alpha,\beta)\frac{\underline{\kappa}_0}{\kappa} \quad (18)$$

and the solution $\underline{T}(\alpha,t)$ can be determined by separation of variables. Similarly, if in (12a) $h$ is replaced by $\underline{h}(\beta)$ such that

$$\underline{h}(\beta) = h\frac{\cosh\alpha_0-\cos\beta}{\cosh\alpha_0+1} \leqslant h, \quad (19)$$

then this boundary condition for the approximate solution $\underline{T}(\alpha,t)$ becomes

$$-(\cosh\alpha_0+1)\frac{\partial \underline{T}(\alpha_0,t)}{\partial\alpha} = \frac{hc}{k}[\underline{T}(\alpha_0,t)-1], \quad (20)$$

which can be satisfied in a simple manner. The plane-surface boundary condition and the initial condition can be exactly satisfied, i.e.

$$\underline{T}(\alpha,0) = 0 \quad (21)$$

and

$$\underline{T}(0,t) = 0. \quad (22)$$

In the approximation (16) $\kappa(\alpha,\beta)$ is replaced by a lower thermal diffusivity and in (18) $h$ is replaced by a lower heat-transfer coefficient $\underline{h}(\beta)$ at the source. Physically, both these changes in parameter would lead to a lower heat flux and a lower temperature distribution. Thus it may be expected that

$$\underline{T}(\alpha,t) \leqslant T(\alpha,\beta,t). \quad (23)$$

A more rigorous proof of the inequality (23) is obtained by applying Lemma I, as shown below.

If $P$ is defined as the parabolic operator

$$P \equiv \frac{\kappa_0}{\kappa}g(\alpha,\beta)\frac{\partial}{\partial t} - \left[\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}\right], \quad (24)$$

then

$$P[\underline{T}] \equiv \frac{\kappa_0}{\kappa}g(\alpha,\beta)\frac{\partial \underline{T}}{\partial t} - \frac{\partial^2 \underline{T}}{\partial\alpha^2}. \quad (25)$$

Combining (17) and (25) gives

$$P[\underline{T}] = \frac{\partial \underline{T}}{\partial t}\left[g(\alpha,\beta)\frac{\kappa}{\kappa_0} - \bar{f}\right]. \quad (26)$$

By assuming that $\partial\underline{T}/\partial t \geqslant 0$ and making use of (18) it can be seen that

$$P[\underline{T}] \leqslant 0, \quad (27)$$

and since by definition

$$P[T] = 0, \quad (28)$$

$$P[\underline{T}] \leqslant P[T]. \quad (29)$$

If $B$ is defined as the "boundary" operator

$$B[\phi] = \begin{cases} \phi(\alpha, t) \text{ if } t = 0, \\ \phi(\alpha, t) \text{ if } \alpha = 0, \\ \left| \dfrac{\partial \phi(\alpha, t)}{\partial \alpha} + \dfrac{hc}{k} \dfrac{1}{(\cosh \alpha - \cos \beta)} \phi(\alpha, t) \right| \\ \qquad \text{if } \alpha = \alpha_0, \end{cases} \quad (30)$$

then the "boundary" conditions (10)–(12) may be written as

$$B[T] = \begin{cases} 0 \text{ if } t = 0, \\ 0 \text{ if } \alpha = 0, \\ \dfrac{hc}{k} \dfrac{1}{(\cos \alpha - \cos \beta)} T \text{ if } \alpha = \alpha_0. \end{cases} \quad (31)$$

However, by making use of the approximate boundary condition (20) it can be seen that

$$B[T] = \begin{cases} 0 \text{ if } t = 0, \\ 0 \text{ if } \alpha = 0, \\ \dfrac{hc}{k} \left[ \dfrac{1}{\cosh \alpha - \cos \beta} - \dfrac{1}{\cosh \alpha + 1} \right] \underline{T}(\alpha, t) \\ \quad + \dfrac{hc}{k} \dfrac{1}{\cosh \alpha + 1} \text{ if } \alpha = \alpha_0. \end{cases} \quad (32)$$

By assuming that $\underline{T}(\alpha_0, t) \leqslant 1$, it is not difficult to show that

$$\frac{hc}{k} \left[ \frac{1}{\cosh \alpha_0 - \cos \beta} - \frac{1}{\cosh \alpha_0 + 1} \right] \times [\underline{T}(\alpha_0, t) - 1] \leqslant 0, \quad (33)$$

i.e.

$$\frac{hc}{k} \left| \frac{1}{\cosh \alpha_0 - \cos \beta} - \frac{1}{\cosh \alpha_0 + 1} \right| \underline{T}(\alpha_0, t) + \frac{hc}{k} \frac{1}{\cosh \alpha_0 + 1} \leqslant \frac{hc}{k} \frac{1}{\cosh \alpha_0 - \cos \beta}. \quad (34)$$

By comparing (31) and (32) and making use of (34) it follows that

$$B[\underline{T}] \leqslant B[T]. \quad (35)$$

Hence, by Lemma I

$$\underline{T}(\alpha, t) \leqslant T(\alpha, \beta, t). \quad (36)$$

The assumptions that $\partial \underline{T}(\alpha, t)/\partial t \geqslant 0$ and $\underline{T}(\alpha_0, t) \leqslant 1$ may be justified after $\underline{T}(\alpha, t)$ is found.

In a similar manner it can be shown that if $g\kappa_0/\kappa$ is replaced by $\underline{f}(\alpha)$ such that $\underline{f}(\alpha) \leqslant g\bar{\kappa}_0/\kappa$ [where $\bar{\kappa}_0$ is the maximum value of $\kappa(\alpha, \beta)$] and $h$ is replaced by $\bar{h}(\beta)$ such that $h \leqslant \bar{h}(\beta)$ then an upper bound $\overline{T}(\alpha, t) \geqslant T(\alpha, \beta, t)$ is found. It should be noted that since a new scale $\bar{\kappa}_0$ is used for $\overline{T}$, the dimensionless time variable $t$ is also changed, i.e. in $\overline{T}(\alpha, t)$

$$t = t^* \bar{\kappa}_0 / c^2. \quad (37)$$

## 5. SOLUTIONS FOR APPROXIMATE FORMULATIONS

$\underline{T}(\alpha, t)$ can be determined from equations (18), (20)–(22) if a suitable expression for $f(\alpha)$ is found, for example,

$$g\underline{\kappa}_0/\kappa \leqslant \frac{1}{(\cosh \alpha - 1)^2} \leqslant \frac{4}{\alpha^4}. \quad (38)$$

By using the right side of the above inequality as $f(\alpha)$ the differential equation for $\underline{T}(\alpha, t)$ becomes

$$\frac{4}{\alpha^4} \frac{\partial \underline{T}}{\partial t} - \frac{\partial^2 \underline{T}}{\partial \alpha^2} = 0, \quad (39)$$

whose solution for the initial condition (21) and the boundary conditions (20) and (22) is found by the method of Laplace transform as

$$\underline{T}(\alpha, t) = \frac{\alpha}{q} \{ -\exp[pr + p^2 t] \times \operatorname{erfc}[pt^{1/2} + \tfrac{1}{2} r t^{-1/2}] + \operatorname{erfc}[\tfrac{1}{2} r t^{-1/2}] \} \quad (40)$$

where

$$p = \frac{\alpha_0 + \dfrac{k}{hc}(\cosh \alpha_0 + 1)}{\dfrac{2k}{hc} \dfrac{(\cosh \alpha_0 + 1)}{\alpha_0^2}}, \quad (41)$$

$$q = \alpha_0 + \frac{k}{hc}(\cosh \alpha_0 + 1), \quad (42)$$

and

$$r = 2\left(\frac{1}{\alpha} - \frac{1}{\alpha_0}\right). \quad (43)$$

It is not difficult to verify that $\underline{T}(\alpha_0, t) \leqslant 1$ and $\partial \underline{T}(\alpha, t)/\partial t \geqslant 0$ as assumed.

An upper bound to the exact solution $\overline{T}(\alpha, t) \geqslant T(\alpha, \beta, t)$ is found using

$$\bar{h}(\beta) = h \left[ \frac{\cos \alpha_0 - \cos \beta}{\cosh \alpha_0 - 1} \right] \quad (44)$$

$$\underline{f}(\alpha) = \frac{1}{4(1 + b\alpha)^2} \leqslant g \frac{\bar{\kappa}}{\kappa}, \quad (45)$$

where $\bar{\kappa}_0$ is the maximum value of $\kappa(\alpha, \beta)$ and

$$b = \frac{\cosh \alpha_0 - 1}{2\alpha_0} = \frac{\sinh^2 \frac{1}{2}\alpha_0}{\alpha_0}. \quad (46)$$

The resulting set of equations for $\overline{T}(\alpha, t)$ becomes

$$\frac{1}{4} \left( \frac{1}{1 + b\alpha} \right)^2 \frac{\partial \overline{T}}{\partial t} - \frac{\partial^2 \overline{T}}{\partial \alpha^2} = 0, \quad (47)$$

$$\overline{T}(\alpha, 0) = 0, \quad (48)$$

$$\overline{T}(0, t) = 0, \quad (49)$$

$$\frac{\partial \overline{T}(\alpha_0, t)}{\partial \alpha} + \frac{hc}{k(\cosh \alpha_0 - 1)} [\overline{T}(\alpha_0, t) - 1] = 0. \quad (50)$$

Upon solving this set by the separation of variables $\overline{T}(\alpha, t)$ is found in the following explicit form.

$$\overline{T}(\alpha, t) = \frac{1}{\left(1 + \dfrac{2bk}{hc}\right) b\alpha_0} \left\{ b\alpha - 16 \sum_{n=1}^{\infty} A_n (1 + b\alpha)^{1/2} \sin[\lambda_n \log(1 + b\alpha)] \exp[-(4\lambda_n^2 + 1) b^2 t] \right\} \quad (51)$$

where

$$A_n = \frac{\lambda_n[\cosh(u_0/2)\sin(\lambda_n u_0) - 2\lambda_n \sinh(u_0/2)\cos(\lambda_n u_0)]}{(4\lambda_n^2+1)[2\lambda_n u_0 - \sin(2\lambda_n u_0)]}, \tag{52}$$

$\{\lambda_n\}$ is the set of roots of the transcendental equation

$$\left(1 + b\alpha_0 + \frac{b^2 k}{hc}\alpha_0\right)\tan[\lambda_n \log(1+b\alpha_0)] = -\frac{2b^2 k}{hc}\lambda_n, \tag{53}$$

and

$$u_0 = \log(1+b\alpha_0). \tag{54}$$

For the special case when $h \to \infty$, the boundary condition (12) on the cylindrical surface takes the isothermal form used by Ioff [4] $T(\alpha_0, \beta, t) = 1$. In this case the above upper and lower bounds become

$$\underline{T}(\alpha, t) = \frac{\alpha}{\alpha_0}\operatorname{erfc}\left[t^{-1/2}\left(\frac{1}{\alpha} - \frac{1}{\alpha_0}\right)\right] \tag{55}$$

and

$$\overline{T}(\alpha, t) = \frac{\alpha}{\alpha_0} + 2\left[\frac{1 + \frac{\alpha}{\alpha_0}\sinh^2\frac{1}{2}\alpha_0}{1+\sinh^2\frac{1}{2}\alpha_0}\right]^{1/2} \sum_{n=1}^{\infty} \frac{(-1)^n(n\pi)}{(n\pi)^2 + [\log(\cosh\frac{1}{2}\alpha_0)]^2}$$
$$\times \sin\left[\frac{n\pi \log\left(1 + \frac{\alpha}{\alpha_0}\sinh^2\frac{1}{2}\alpha_0\right)}{\log(1+\sinh^2\frac{1}{2}\alpha_0)}\right] \exp\left[-\left\{\frac{(n\pi)^2}{[\log(\cosh\frac{1}{2}\alpha_0)]^2} + 1\right\}\frac{\sinh^4\frac{1}{2}\alpha_0}{\alpha_0^2}\cdot t\right] \tag{56}$$

Both of these exactly satisfy the boundary condition on the cylindrical surface. Solutions in which $\underline{\kappa}_0 = \bar{\kappa}_0 = \kappa_0$ are presented in Figs. 3 and 4 as functions of $(\alpha/\alpha_0)$ for different values of $\alpha_0$ and Fourier number $(\kappa_0 t^*/c^2)$.

The overall transient resistance is of interest in the design of a buried heat-source system and can be estimated using the solutions (55) and (56) for the isothermal boundary condition on the pipe and constant thermal diffusivity. The expression for the re-

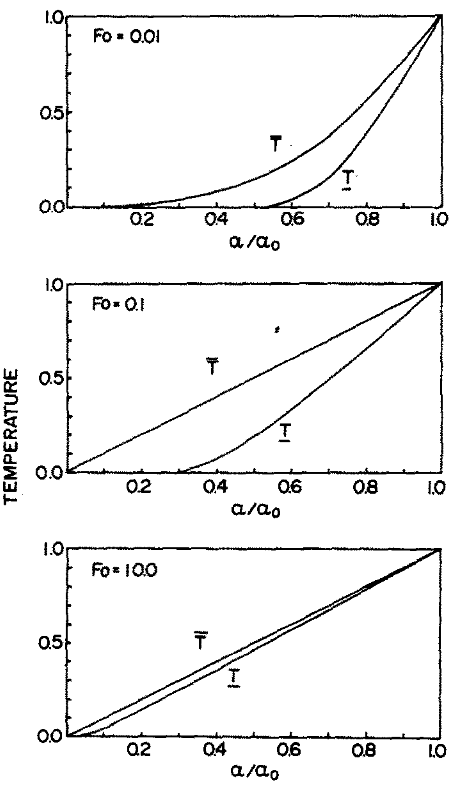


FIG. 3. Temperature distribution at various values of Fourier number for $\alpha_0 = 1$.

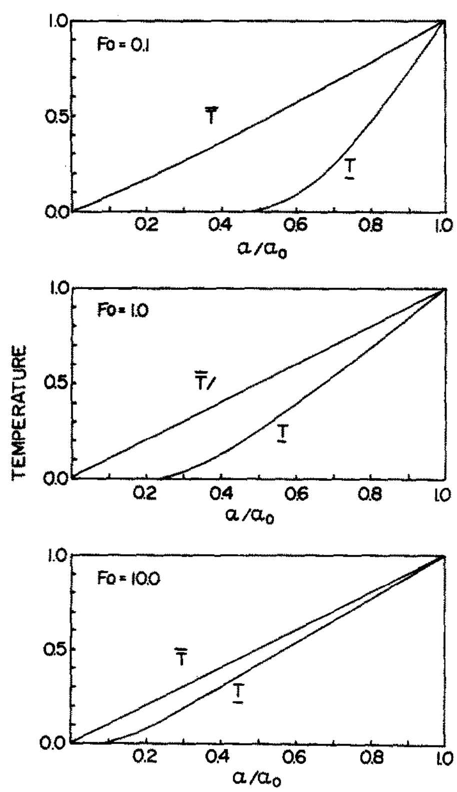


FIG. 4. Temperature distribution at various values of Fourier number for $\alpha_0 = 2$.

sistance is defined in terms of the source temperature $T_0^*$ as

$$R = T_0^*/Q \tag{57}$$

where $Q$ is the total heat flow rate from the source and is given by

$$Q = 2\pi k \cdot \frac{\rho}{c} \cdot \cosh\alpha_0 \left.\frac{\partial T^*}{\partial \alpha}\right|_{\alpha=\alpha_0}. \tag{58}$$

It is convenient to nondimensionalize $R$ with the steady-state resistance [1]

$$R_\infty = \alpha_0/2\pi k, \tag{59}$$

i.e.

$$\frac{R}{R_\infty} = \frac{\tanh\alpha_0}{\alpha_0} \left.\frac{\partial T}{\partial \alpha}\right|_{\alpha=\alpha_0}. \tag{60}$$

As is clear from Figs. 3 and 4 the lower-bound solution gives an upper-bound value for the temperature gradient at $\alpha = \alpha_0$ while the upper-bound solution gives a lower bound on the gradient at that position. Hence, upper and lower bounds on the overall transient thermal resistance may be derived from (56) and (55), respectively. The results for several values of $d/\rho$ are presented in Fig. 5 in terms of the steady-state resistance. As can be expected

$$\lim_{\substack{F_0 \to \infty \\ d/\rho \to \infty}} R/R_\infty \to 1. \tag{61}$$

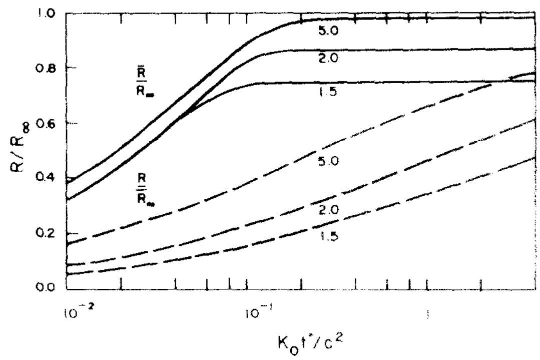


FIG. 5. Bounds on the transient thermal resistance for various $d/\rho$.

## 6. DISCUSSION

In order to obtain the upper and lower bounds to the unknown exact transient solution to the heat-conduction equation in the region surrounding a buried cylindrical heat source the dependence of the temperature on the space variable $\beta$ was eliminated. It can be expected that the exact temperature distribution is a maximum on $\beta = \pi$ for a given value of $\alpha$ and, hence, the upper bound $\bar{T}(\alpha, t)$ should be best along $\beta = \pi$, i.e. along the $x$-axis between the cylinder and the plane surface (see Fig. 1). Since the maximum heat flux is also along this portion of the $x$-axis, the solutions (53) and (58) should be useful to engineers for predicting upper bounds on the transient heat flux.

The lower bound is less useful since the simplification (39) is only reasonable along $\beta = 0$, i.e. below the cylinder and for small depth relative to the cylinder radius. However, either bound may be used to estimate the time to closely achieve steady state for situations in which the isothermal boundary condition is valid since the approximate solutions approach the exact steady-state solutions for large time.

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## REFERENCES


1. E. R. G. Eckert and R. M. Drake, *Heat and Mass Transfer*, 2nd edn, pp. 60–64. McGraw-Hill, New York (1959).
2. R. Thiyagarajan and M. M. Yovanovich, Thermal resistance of a buried cylinder with constant flux boundary condition, *J. Heat Transfer* **96C**, 249–250 (1974).
3. I. A. Ioff, Plane nonstationary heat conduction problem for a semi-infinite body with an internal isothermal cylindrical heat source, *Zh. Tekh. Fiz.* **29**, 417 [English Translation: *J. Tech. Phys.* **29**, 369–374] (1959).
4. I. A. Ioff, A problem of transient heat conduction in a semi-bounded body with an internal cylindrical heat source, *Inzh. Fiz. Zh.* **23**, 351–355 (1972) [English Translation: *J. Engng Phys.* **23**, 1051–1054] (1972).
5. E. Adams, Theoretical estimate of temperature change in a waterway polluted by addition of hot water, Stability, p. 423. Study No. 6, Solid Mechanics Division, University of Waterloo (1972).
6. S. S. Sadhal and W. W. Martin, Heat transfer through drop condensate using differential inequalities, *Int. J. Heat Mass Transfer* **20**, 1401–1407 (1977).
7. W. Walter, *Differential and Integral Inequalities*. Springer, Berlin (1970).
8. S. S. Sadhal, The application of differential inequalities to the heat conduction equation in complex geometries, M.A.Sc. thesis, Dept. of Mech. Engng, University of Toronto (1976).


## LIMITES DE LA DISTRIBUTION TRANSITOIRE DE TEMPERATURE DUE A UNE SOURCE DE CHALEUR ENTERREE ET CYLINDRIQUE

**Résumé**—La théorie des inégalités différentielles est appliquée pour déterminer les limites supérieures et inférieures sur la distribution transitoire de température dans la région qui entoure une source de chaleur cylindrique enterrée, à section droite circulaire. On obtient des solutions pour une condition de convection sur la surface cylindrique et de température constante sur la surface libre, avec la profondeur pour paramètre. L'utilisation des solutions approchées pour les estimations industrielles du transfert thermique transitoire dans le cas des canalisations enterrées est discutée brièvement.

## GRENZEN DER INSTATIONÄREN TEMPERATURVERTEILUNG AUFGRUND EINER VERGRABENEN ZYLINDRISCHEN WÄRMEQUELLE

**Zusammenfassung**—Mit Hilfe der Theorie der Differential-Ungleichungen werden die obere und untere Grenze der instationären Temperaturverteilung in dem Gebiet um eine vergrabene kreiszylindrische Wärmequelle bestimmt. Lösungen wurden erhalten für eine Wärmeübergangsrandbedingung auf der zylindrischen und konstante Temperatur an der freien Oberfläche mit der Tiefe als Parameter. Die Anwendung der Näherungslösungen auf die ingenieurmäßige Abschätzung der instationären Wärmeabgabe unterirdisch verlegter Rohrleitungen wird kurz erörtert.

## ГРАНИЦЫ НЕСТАЦИОНАРНОГО РАСПРЕДЕЛЕНИЯ ТЕМПЕРАТУРЫ ВСЛЕДСТВИЕ ЗАГЛУБЛЕНИЯ ЦИЛИНДРИЧЕСКОГО ИСТОЧНИКА ТЕПЛА

**Аннотация** — Теория дифференциальных неравенств используется для определения верхней и нижней границ нестационарного распределения температуры вокруг заглубленного цилиндрического источника тепла круглого сечения. Решения получены для конвективного граничного условия на поверхности цилиндра и постоянной температуры на свободной поверхности с использованием глубины расположения источника в качестве параметра. Дан краткий анализ применения приближенных решений для инженерных расчётов нестационарного переноса тепла от заглубленных трубопроводов.